stochastic processes and their

Multifractal analysis of the occupation measures of a kind of stochastic processes

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Abstract

Let $\{X(t),\ 0\leqslant t\leqslant 1\}$ be a stochastic process whose range is a random Cantor-like set depending on an α-sequence $(0<\alpha<1)$ and μ is the occupation measure of $X(t)$. In this paper we examine the multifractal structure of μ and obtain the fractal dimensions of the sets of points of where the local dimension of μ is different from α. It is interesting to notice that the final results of this paper are identical to those for the occupation measure of a stable subordinator with index α, yet the stochastic process under consideration in this work is not even a Markov process.

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1. Introduction

Given $(\Omega,\mathscr{F},P):=([0,1]^{\mathbb{N}},\mathscr{B}([0,1])^{\mathbb{N}},\mathscr{L}^{\mathbb{N}})$, $\mathscr{L}$ is the Lebesgue measure on $[0,1]$. Let $\{\xi_i\}_{i\geqslant 1}$ be a sequence of i.i.d. random variables on the above probability space satisfying $\xi_i(\omega)=\omega_i$, where $\omega=(\omega_1,\omega_2,\ldots,\omega_i,\ldots)$, $\omega_i\in[0,1]$ and $\{a_i\}_{i\geqslant 1}$ be an α-sequence $(0<\alpha<1)$ of positive numbers in the sense that

$$\sum_{i\geqslant 1}a_i=1,\ a_i\downarrow 0,\ \sum_{i\geqslant 1}\min(a_i,\varepsilon)\approx\varepsilon^{1-\alpha},$$
$$\sup\{k:\ a_k\geqslant\varepsilon\}=\sup\{k:\ a_k>\varepsilon\}\approx\varepsilon^{-\alpha}. \tag{1.1}$$

$f(x)\approx g(x)$ on D means that there exist two constants $c_1,c_2>0$ such that

$$c_1g(x)\leqslant f(x)\leqslant c_2g(x),\quad \forall x\in D.$$

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Now define

$$X(t)=\sum_{i\geqslant 1} a_i I_{\{0\leqslant \xi_i < t\}}, \quad \mu(A)=\mathscr{L}(\{0\leqslant t\leqslant 1\colon X(t)\in A\}). \tag{1.2}$$

This occupation measure has been used to find the Hausdorff dimension and the Hausdorff and packing measures of a random Cantor-like set depending on the above α-sequence, see Hawkes (1984), Hu (1995 and 1996). From some known results (Hu, 1995, 1996) we have that for a.s. ω,

$$d(\mu,x):=\lim_{r\downarrow 0}\frac{\log \mu B(x,r)}{\log r}=\alpha. \tag{1.3}$$

In this paper our aim is to examine the structure of the set $\{x\in \operatorname{supp}\mu\colon d(\mu,x)\neq\alpha\}$.

Recently, multifractal analysis has been proved to be a useful tool in the analysis of singular measures, both in theory and applications, see, for example, Falconer (1990), Halsey et al. (1986). Let

$$A_{\beta,m}=\{x\in \operatorname{supp} m\colon d(m,x)=\beta\},$$

where m is an arbitrary probability measure, and

$$f_m(\beta)=\dim(A_{\beta,m}), \qquad F_m(\beta)=\operatorname{Dim}(A_{\beta,m}),$$

where dim and Dim denote Hausdorff and packing dimensions respectively. $\{A_{\beta,m}\}_{\beta\geqslant 0}$ can be thought of as a multifractal decomposition of supp m. Actually an elaborate formalism has been developed to determine $f_m(\beta)$ and $F_m(\beta)$, see Olsen (1995), Peyrière (1992) for example. This formalism works very well for a few types of measures for which the spectrum function $f_m(\beta)$ has been rigorously determined and it turns out that there exists $\alpha_1<\alpha_2$ such that $f_m(\beta)=0$ on $[0,\infty)-[\alpha_1,\alpha_2]$ and $f_m(\beta)$ is concave and smooth on (α_1,α_2), see Falconer (1994), Olsen (1995). Unfortunately, the formalism breaks down for some very interesting measures, for example, the stable occupation measures, see Hu and Taylor (1997). In this work we are going to prove that for a.s. ω,

$$A_{\beta,\mu}=\emptyset \quad \text{for } \beta\neq\alpha,\ \alpha \text{ is as defined in Eq. (1.1).}$$

Therefore the function $\tau(\beta)$ (see Olsen, 1995; Peyrière, 1992) has no definition here. Thus the formalism also breaks down in this case.

Section 2 gives the preliminary definitions and lemmas. In Section 3 we discuss the dimensions of the set of points where the lower local dimension is α. We prove that for a.s. ω, this set is the support of μ, which implies that the multifractal spectrum of this set is trivial. In Section 4 we examine the set of points where the upper local dimension is equal to β ($\alpha\leqslant\beta\leqslant 2\alpha$), say E_β, and obtain its lower bound of Hausdorff dimension which is $(2\alpha-\beta)/\gamma$ ($\gamma=\beta/\alpha$), while the lower bound of the packing dimension is $2\alpha-\beta$, α and β are the same as above. The ordinary method of calculating dimensions fails here, so we first construct a Cantor like set contained in $C_\beta\equiv\{x\in\operatorname{supp}\mu$: the upper local dimension of x is greater than or equal to $\beta\}$ and then obtain the lower bound of Hausdorff dimension of this set. An easy argument gives the corresponding result for E_β. In Section 5 we find out when $\beta>2\alpha$, for a.s. ω, for any point $x\in\operatorname{supp}\mu$, the upper local dimension of x is less than β. We also obtain the upper bound of Hausdorff dimension of E_β ($\alpha \leqslant \beta \leqslant 2\alpha$) which is the same as the lower bound, this implies E_β is not dimension regular.

It is interesting to notice that though the stochastic process under consideration in this work is far from a stable process, the final results of this paper are identical to those for the occupation measure of a stable subordinator with index α (see Hu and Taylor, 1997). The proofs present more difficulties because the process is lack of independence and Markov property.

2. Preliminaries

Let us recall the definition of the random compact sets obtained by randomly ordering the intervals complementary to $[0,1]$ which was first given in Hawkes (1984). From now on to the end of this paper we always denote the α-sequence by $\{a_i\}_{i\geqslant 1}$.

Definition 2.1. Suppose $(\Omega, \mathscr{F}, P) = ([0,1], \mathscr{B}([0,1]), \mathscr{L})^{\mathbb{N}}$ ($\mathscr{L}$ is the Lebesgue measure on $[0,1]$, $\mathbb{N}$ is the set of positive integers), and $\{\xi_n\}_{n\geqslant 1}$ is a sequence of i.i.d. random variables satisfying $\xi_i(\omega) = \omega_i$, where $\omega = (\omega_1, \omega_2, \ldots)$, $\omega_i \in [0,1]$. Let $Q_n(\omega) = \{m\colon \omega_m < \omega_n\}$ and $l_n(\omega) = \sum_{s\in Q_n(\omega)} a_s$. Define

$$K(\omega) = [0,1] - \bigcup_{n\geqslant 1} J_n(\omega), \qquad J_n(\omega) = (l_n(\omega), l_n(\omega) + a_n). \tag{2.1}$$

$K(\omega)$ is said to be a random compact set belonging to the sequence $\{a_i\}_{i\geqslant 1}$ and can be thought of as the set derived by giving the complementary intervals in random order on the line. When $a_i = 1/3^n$, $i = 2^{n-1} + k$, $k = 0, \ldots, 2^{n-1} - 1$, $n = 1, 2, \ldots$, $\{a_i\}_{i\geqslant 1}$ is a $\log 2/\log 3$-sequence and $K(\omega)$ is said to be a random re-ordering of the Cantor set.

Let

$$v(A) = v(A, \omega) = \sum_{i\geqslant 1} a_i I_{\{\xi_i \in A\}}(\omega). \tag{2.2}$$

By Eq. (1.2) one can see that $X(t) = v([0,t))$ and $\mu(A) = \mathscr{L}(\{0 \leqslant t \leqslant 1\colon v([0,t)) \in A\})$. Furthermore $\overline{X([0,1])} = K(\omega)(X(E) = \{X(t)\colon t \in E\})$ and $\operatorname{supp}\mu = K(\omega)$.

Definition 2.2. Let m be a probability measure on $\mathscr{B}(\mathbb{R})^d$, define

$$\underline{d}(m,x) = \liminf_{r\downarrow 0} \frac{\log m(B(x,r))}{\log r},$$

$$\bar{d}(m,x) = \limsup_{r\downarrow 0} \frac{\log m(B(x,r))}{\log r}, \quad x \in \mathbb{R}^d,$$

where $B(x,r) = \{y \in \mathbb{R}^d\colon |y-x| < r\}$. When $\underline{d}(m,x) = \bar{d}(m,x)$ a.e. $[m]$, we say m is a fractal measure (or dimension regular) and denote the common value as $d(m,x)$. $\underline{d}(m,x)$ and $\bar{d}(m,x)$ are called the lower and upper local dimensions, respectively.

The proofs of the following two lemmas are almost the same as those of Lemmas 1.2 and 1.4 in Hu (1995). Since we will use them repeatedly later, we state them here for the convenience of the reader.

Lemma 2.1. *Let* v *be defined as in* Eq. (2.2), $\lambda > 1$, *there exists* h_0 *such that*

$$P(v(J) > \lambda h^{1/\alpha}) \approx \lambda^{-\alpha}, \quad J = [0, h),\ h \leqslant h_0,\ \lambda \geqslant \lambda_0.$$

Proof. Let $q(t) = \max\{m: a_m \geqslant t\}$. Since

$$\begin{aligned} P(v(J) > \lambda h^{1/\alpha}) &\geqslant P\left(\sum_{a_i \geqslant \lambda h^{1/\alpha}} a_i I_{\{0 \leqslant \xi_i < h\}} > \lambda h^{1/\alpha}\right) \\ &= P(\exists i \leqslant q(\lambda h^{1/\alpha}) \text{ such that } I_{\{0 \leqslant \xi_i < h\}} = 1) \\ &= 1 - (1-h)^{q(\lambda h^{1/\alpha})}, \end{aligned}$$

and by Eq. (1.1) we know

$$c_1 \lambda^{-\alpha} h^{-1} \leqslant q(\lambda h^{1/\alpha}) \leqslant c_2 \lambda^{-\alpha} h^{-1},$$

where c_1 and c_2 are two positive constants. It follows that, if h is small enough and λ is larger enough,

$$\begin{aligned} P(v(J) > \lambda h^{1/\alpha}) &\geqslant 1 - (1-h)^{c_1 \lambda^{-\alpha} h^{-1}} \\ &\geqslant 1 - \exp(-c_1 \lambda^{-\alpha}) \geqslant \tfrac{1}{2} c_1 \lambda^{-\alpha}. \end{aligned}$$

(Note: $e^{-2x} \leqslant 1 - x \leqslant e^{-x}$ if $0 < 2x < 1$.)

For the other direction, let $Z_i = a_i I_{\{0 \leqslant \xi_i < h\}}$, it is clear that

$$P(v(J) > \lambda h^{1/\alpha}) \leqslant P\left(\sum_{a_i < s^{-1}} Z_i > \frac{\lambda}{2} h^{1/\alpha}\right) + P\left(\sum_{a_i \geqslant s^{-1}} Z_i > \frac{\lambda}{2} h^{1/\alpha}\right).$$

Let $s = 2h^{-1/\alpha}\lambda^{-1}$, then

$$\begin{aligned} &P\left(\sum_{a_i \geqslant s^{-1}} Z_i > \frac{\lambda}{2} h^{1/\alpha}\right) \\ &\quad \leqslant P(\exists i \leqslant q(s^{-1}) \text{ such that } I_{\{0 \leqslant \xi_i < h\}} = 1) \\ &\quad = 1 - (1-h)^{q(s^{-1})} = 1 - (1-h)^{q((1/2)h^{1/\alpha}\lambda)} \\ &\quad \leqslant 1 - (1-h)^{2^\alpha c_2 h^{-1} \lambda^{-\alpha}} \leqslant 1 - \exp(-2c_2 \lambda^{-\alpha}) \leqslant 2c_2 \lambda^{-\alpha}. \end{aligned}$$

When λ is large enough and h is small enough. But

$$P\left(\sum_{a_i < s^{-1}} Z_i > \frac{\lambda}{2} h^{1/\alpha}\right) \leqslant \exp\left(-t \frac{\lambda}{2} h^{1/\alpha}\right) E \exp\left(t \sum_{a_i < s^{-1}} Z_i\right)$$

and

$$\begin{aligned} E \exp\left(t \sum_{a_i < s^{-1}} Z_i\right) &= \prod_{a_i < s^{-1}} [1 + h(e^{t a_i} - 1)] \\ &\leqslant \exp\left(h \sum_{a_i < s^{-1}} (e^{t a_i} - 1)\right) \quad (1 + x < e^x \text{ for all } x > 0) \end{aligned}$$

$$\leqslant \exp\left(2h \sum_{a_i < s^{-1}} ta_i\right) \quad (e^x - 1 \leqslant 2x, \text{ if } 0 < x < 1)$$

$$\leqslant \exp(c_3 hts^{\alpha-1}), \quad c_3 > 0 \text{ is a constant.}$$

Let $t = h^{-1/\alpha}$ and $s = 2h^{-1/\alpha}\lambda^{-1}$; then we obtain

$$P\left(\sum_{a_i < s^{-1}} Z_i > \frac{\lambda}{2} h^{1/\alpha}\right) \leqslant \exp\left(-\frac{1}{2} t\lambda h^{1/\alpha} + c_3 hts^{\alpha-1}\right)$$

$$= \exp\left(-\frac{1}{2}\lambda + c_3 \lambda^{1-\alpha}\right).$$

Thus when λ is large enough we have

$$P(v(J) > \lambda h^{1/\alpha}) \leqslant c_4 \lambda^{-\alpha}, \quad c_4 > 0 \text{ is a constant.}$$

We have proved the lemma. □

Lemma 2.2. *Let N_n be the number of dyadic intervals with length 2^{-n} which have non-empty intersections with $K(\omega)$; then*

$$N_n \approx 2^{n\alpha}.$$

Proof. It is clear that

$$N_n = 2^n - \sum_{j=1}^{2^n} \sum_{k=1}^{q_n} I_{\{I_{j,n} \subset J_k\}}$$

$$= 2^n - \sum_{k=1}^{q_n} \sum_{j=1}^{2^n} I_{\{I_{j,n} \subset J_k\}},$$

where $J_k = (t_k, t_k + a_k)$, $t_k = \sum_{i \in Q_k} a_i$, $Q_k = \{m: \omega_m < \omega_k\}$ and $q_n = \max\{m: a_m \geqslant 1/2^n\}$.

But

$$2^n a_k - 2 \leqslant \sum_{j=1}^{2^n} I_{\{I_{j,n} \subset J_k\}} \leqslant 2^n a_k$$

and by Eq. (1.1) we have

$$\sum_{k=1}^{q_n} 2^n a_k = 2^n \sum_{a_k \geqslant 2^{-n}} a_k = 2^n \left[1 - \sum_{a_k < 2^{-n}} a_k\right] \leqslant 2^n [1 - c_1 2^{-n(1-\alpha)}],$$

$$\sum_{k=1}^{q_n} (2^n a_k - 2) = 2^n \sum_{a_k \geqslant 2^{-n}} a_k - 2q_n \geqslant 2^n (1 - c_2 2^{-n(1-\alpha)}) - c_3 2^{n\alpha},$$

where $c_1 > 0, c_2 > 0$ and $c_3 > 0$ are constants. Thus

$$N_n \approx 2^{n\alpha}. \quad \square$$

We will use two-dimensional indices: they are Hausdorff dimension $\dim(\cdot)$ and packing dimension $\mathrm{Dim}(\cdot)$. Readers can find their definitions and properties in Taylor (1987). For a subset $A \subset \mathbb{R}^d$, note that

$$0 \leqslant \dim(A) \leqslant \mathrm{Dim}(A) \leqslant d. \tag{2.3}$$

We denote $\phi - m$ and $\phi - p$ as the Hausdorff measure and packing measure derived by ϕ, respectively.

Recently, Perkins and Taylor (1998) introduced the notion of a γ-thin set $A \subset \mathbb{R}^d$. Let $\gamma > 1$, $x \in \mathbb{R}^d$, we say A is γ-thin at x if there exists a sequence $\{r_i\}_{i \geqslant 1}$ decreasing to zero such that

$$[B(x, r_i) - B(x, r_i^{\gamma})] \cap A = \emptyset.$$

Definition 2.3. The set $A \subset \mathbb{R}^d$ is said to be γ-thin if A is γ-thin at x, $\forall x \in A$.

The following lemma is from Perkins and Taylor (1998).

Lemma 2.3. *Suppose $A \subset \mathbb{R}$ is γ-thin, then* $\mathrm{Dim}(A) \geqslant \gamma \dim(A)$.

3. Dimensions of the set of points where its lower local dimension is α

By Theorem 2.1 in Hu (1995), for a fixed $t \in [0,1]$, it is true that

$$c_1 \leqslant \limsup_{r \downarrow 0} \frac{\mu(x - r, x + r)}{r^{\alpha} (\log\log 1/r)^{1-\alpha}} \leqslant c_2 \ \text{ a.s.,}$$

where $x = v[0,t)$, $c_1, c_2 > 0$ are constants. This tells us that $\underline{d}(\mu, x) = \alpha$ a.s.

On the other hand, using Theorem 1.1 in Hu (1996) we obtain

$$\liminf_{r \downarrow 0} \frac{\mu(x - r, x + r)}{r^{\alpha}} \left(\log \frac{1}{r}\right)^{1/2} = 0 \ \text{ a.s.}$$

and

$$\liminf_{r \downarrow 0} \frac{\mu(x - r, x + r)}{r^{\alpha}} \left(\log \frac{1}{r}\right)^{1/2+\varepsilon} = \infty \ \text{ a.s.,}$$

where $\varepsilon > 0$ is a constant. The above results immediately give $\bar{d}(\mu, x) = \alpha$ a.s. Finally, we have obtained

$$\forall t \in [0,1], \ \ d(\mu, v[0,t)) = \alpha \ \text{ a.s.} \tag{3.1}$$

Let $Z = \{(\omega, t) \in \Omega \times [0,1] : d(\mu, v[0,t)) = \alpha\}$. Then Eq. (3.1) and Fubini Theorem give

$$\int\!\!\int_Z \mathrm{d}P\, \mathrm{d}t = \int_0^1 \mathrm{d}t \left(\int_{\Omega} I_{Z(t)}\, \mathrm{d}P\right) = 1 \quad \text{where } Z(t) = \{\omega\colon (\omega, t) \in Z\}.$$

Using Fubini Theorem again one obtains

$$\int_{\Omega} \left(\int_0^1 I_{Y(\omega)}\, \mathrm{d}t\right) \mathrm{d}P = 1 \quad \text{where } Y(\omega) = \{t\colon (\omega, t) \in Z\},$$

so $\mathscr{L}(Y(\omega)) = 1$ a.s., which means

$$\mu(\{v[0,t)\colon t \in [0,1], d(\mu, v[0,t)) = \alpha\}) = 1 \text{ a.s.}$$

We have proved the following theorem:

Theorem 3.1. *Let $A_\alpha = \{x \in \operatorname{supp}\mu\colon d(\mu,x) = \alpha\}$; then for a.s. $\omega, \mu(A_\alpha) = 1$.*

Lemma 3.1. *For a.s. ω, there exists $\delta = \delta(\omega) > 0$ such that when $t \leqslant \delta$,*

$$v[0,t+s) - v[0,s) \geqslant ct^{1/\alpha}\left(\log\frac{1}{t}\right)^{-2/\alpha}, \quad \forall s \in [0,1], \tag{3.2}$$

$c > 0$ is a constant.

Proof. It is sufficient to prove that for a.s. ω, there exists an integer $k_0 = k_0(\omega)$ such that when $k \geqslant k_0$,

$$\frac{v[0,s+2^k) - v[0,s)}{(2^{-k})^{1/\alpha}(k\log 2)^{-2/\alpha}} \geqslant 1, \ \ \forall s \in [0,1).$$

Given k, let $t_k = (2^{-k})^{1/\alpha}(k\log 2)^{-2/\alpha}$ and $m_k = \sup\{n\colon a_n \geqslant t_k\}$. One can see that

$$\begin{aligned}
&\{v[0,s+2^{-k}) < t_k\} \\
&\quad\subset \bigcup_{i=0}^{2^k-1}\left(\left\{v\left[\frac{i}{2^k}, \frac{2i+1}{2^{k+1}}\right) < t_k\right\} \cup \left\{v\left[\frac{2i+1}{2^{k+1}}, \frac{i+1}{2^k}\right) < t_k\right\}\right) \\
&\quad\subset \bigcup_{i=0}^{2^k-1}(B_{i,k} \cup C_{i,k}) \equiv H_k,
\end{aligned}$$

where

$$B_{i,k} = \left\{\omega\colon \omega_j \in [0,1] - \left[\frac{i}{2^k}, \frac{2i+1}{2^{k+1}}\right), \ 1 \leqslant j \leqslant m_k\right\},$$

$$C_{i,k} = \left\{\omega\colon \omega_j \in [0,1] - \left[\frac{2i+1}{2^{k+1}}, \frac{i+1}{2^k}\right), \ 1 \leqslant j \leqslant m_k\right\}.$$

Note that $P(B_{i,k}) = P(B_{1,k}) = P(C_{i,k}) = P(C_{1,k}), \ 0 \leqslant i \leqslant 2^k - 1$. Therefore by the independence of $\{\xi_i\}(\xi_i(\omega) = \omega_i)$, we have

$$\begin{aligned}
P(H_k) &\leqslant 2^k(P(B_{1,k}) + P(C_{1,k})) \\
&\leqslant 2^k \cdot 2(1 - 2^{-k-1})^{c_1 2^k (k\log 2)^2} (m_k \approx t_k^{-\alpha}) \\
&\leqslant \frac{c_2}{k^2}, \ \ c_1, c_2 > 0 \text{ are constants.}
\end{aligned}$$

Thus $\sum_{k\geqslant 1} P(A_k) < \infty$. By Borel–Cantelli Lemma we know $P(\limsup_{k\to\infty} H_k) = 0$, which implies that $P(\liminf_{k\to\infty} H_k^c) = 1$, but for any $s \in [0,1]$,

$$H_k^c \subset \left\{\frac{v[0,s+2^{-k}) - v[0,s)}{(2^{-k})^{\frac{1}{\alpha}}(k\log 2)^{-2/\alpha}} \geqslant 1\right\} \equiv W(k,s),$$

so $P(\liminf_{k\to\infty} \bigcap_{s\in[0,1]} W(k,s)) = 1$, which gives Lemma 3.1. $\square$

Corollary 3.1. *For a.s.* $\omega, \underline{d}(\mu,x)\geqslant\alpha$, $\forall x\in\operatorname{supp}\mu$.

Proof. Lemma 3.1 tells us that for a.s. ω, there exists $\delta=\delta(\omega)>0$ such that when $t\leqslant\delta$, we have

$$v[0,s+t)-v[0,s)\geqslant ct^{1/\alpha}\left(\log\frac{1}{t}\right)^{-2/\alpha},\quad \forall s\in[0,1],$$

$c>0$ is a constant. Therefore $\mu(x,x+r)\leqslant c'r^{\alpha}(\log 1/r)^2$ $(\forall x\in\operatorname{supp}\mu)$, when $r\leqslant r_0=r_0(\omega)$ (r_0 depends on $\delta, c'>0$ is a constant). See Taylor and Wendel (1966) for the computation in detail. Similarly $\mu(x-r,x)\leqslant c'r^{\alpha}(\log 1/r)^2$, $\forall x\in\operatorname{supp}\mu$. Finally

$$\log\mu(x-r,x+r)\leqslant\log\left[2c'r^{\alpha}\left(\log\frac{1}{r}\right)^2\right]\leqslant\alpha\log r+\log\left[2c'\left(\log\frac{1}{r}\right)^2\right],$$

thus for a.s. ω, $\underline{d}(\mu,x)\geqslant\alpha$, $\forall x\in\operatorname{supp}\mu$. $\square$

Lemma 3.2. *Given* $\varepsilon>0$, *for a.s.* ω, *we have*

$$\limsup_{r\downarrow 0}\frac{\mu(x-r,x+r)}{r^{\alpha+\varepsilon}}>0,\quad \forall x\in\operatorname{supp}\mu.$$

Proof. First we notice the fact that $\operatorname{supp}\mu=K(\omega)=X([0,1])$, where X is defined as in Eq. (1.1).

Let $Q=\{s_1,s_2,\ldots\}$ be the set of all rational numbers in $[0,1]$. Since for every ω, $X(s)=v[0,s)$ is left continuous, one can see that $\{x_i=v[0,s_i)$: $i=1,2,\ldots\}$ is dense in $\operatorname{supp}\mu$.

By Theorem 2.1 (Hu, 1995), we know there is a subset of Ω, say, Ω_0, such that $P(\Omega_0)=1$ and $\forall\omega\in\Omega_0$,

$$\frac{\mu(x_i,x_i+2^{-k_n^{(i)}})}{(2^{-k_n^{(i)}})^{\alpha+\varepsilon}}\geqslant c_1\quad\text{and}\quad\frac{\mu(x_i-2^{k_n^{(i)}},x_i)}{(2^{-k_n^{(i)}})^{\alpha+\varepsilon}}\geqslant c_1,\quad\forall n,$$

$c_1>0$ is a constant. $\{k_n^{(i)}\}_{n\geqslant1}$ depends on ω, for any i.

$\forall\omega\in\Omega_0$, if there exists $x\in\operatorname{supp}\mu$, such that

$$\limsup_{r\downarrow 0}\frac{\mu(x-r,x+r)}{r^{\alpha+\varepsilon}}=0,$$

then given $0<\delta<\frac{1}{2}c_1/2^{\alpha+\varepsilon}$, there exists $r_0=r_0(\omega,\delta)$ such that for every $r\leqslant r_0$,

$$\mu(x-r,x+r)\leqslant\delta r^{\alpha+\varepsilon}. \tag{3.3}$$

Since $\{x_i$: $i=1,\ldots\}$ is dense in $\operatorname{supp}\mu$, so for every n, $\bigcup_i(x_i-2^{-k_n^{(i)}},x_i+2^{-k_n^{(i)}})$ is a covering of $\operatorname{supp}\mu$. Hence for each n, there exists $i=i(n)$ such that $x\in(x_i-2^{-k_n^{(i)}},x_i+2^{-k_n^{(i)}})$. If $x>x_i$, take $r=x-(x_i-2^{k_n^{(i)}})$, then

$$\begin{aligned}\mu(x-r,x)&\geqslant\mu(x_i-2^{-k_n^{(i)}},x_i)\geqslant c_1(2^{-k_n^{(i)}})^{\alpha+\varepsilon}\\&\geqslant c_1(\tfrac{1}{2}r)^{\alpha+\varepsilon}>2\delta r^{\alpha+\varepsilon};\end{aligned}$$

if $x \leqslant x_i$, take $r = x_i + 2^{-k_n^{(i)}} - x$, then similarly we have $\mu(x, x+r) > 2\delta r^{\alpha+\varepsilon}$. Thus if we take $r = (x_i + 2^{-k_n^{(i)}} - x) \vee (x - (x_i - 2^{-k_n^{(i)}}))$, we always have

$$\mu(x-r, x+r) > 2\delta r^{\alpha+\varepsilon}.$$

This contradicts with Eq. (3.3). Thus $\forall \omega \in \Omega_0$,

$$\limsup_{r \downarrow 0} \frac{\mu(x-r, x+r)}{r^{\alpha+\varepsilon}} > 0, \quad \forall x \in \operatorname{supp} \mu. \qquad \square$$

From this lemma we immediately obtain:

Corollary 3.2. *For a.s.* ω, $\underline{d}(\mu, x) \leqslant \alpha$, $\forall x \in \operatorname{supp} \mu$.

Combining Corollaries 3.1 and 3.2 we finally come to our first main theorem.

Theorem 3.2. *For a.s.* ω, $\underline{d}(\mu, x) = \alpha$, $\forall x \in \operatorname{supp} \mu$.

According to Theorem 2.2 in Hu (1995), Theorem 1.2 in Hu (1996) and Theorem 3.2 we know that for a.s. ω, $B_\alpha \equiv \{x \in \operatorname{supp} \mu : \underline{d}(\mu, x) = \alpha\}$ has exact Hausdorff measure function $\phi(t) = t^\alpha (\log\log 1/t)^{1-\alpha}$ and the integral test for the packing measure of B_α is

$$\psi - p(B_\alpha) = \begin{cases} 0 \\ \infty \end{cases} \text{a.s. according as } \int_{0+} \frac{g^2(t)}{t}\,\mathrm{d}t \begin{cases} < \infty, \\ = \infty, \end{cases}$$

where $\psi(t) = t^\alpha g(t)$, $g(t)$ is a measure function. It is easy to see that $\dim(B_\alpha) = \operatorname{Dim}(B_\alpha) = \alpha$ a.s., which means B_α is a.s. dimension regular.

4. The lower bound of Hausdorff dimension of the set of points where upper local dimensions equal to β, $\alpha \leqslant \beta \leqslant 2\alpha$

Our object in this section is to examine the fractal property of $C_\beta \equiv \{x \in \operatorname{supp} \mu : \bar{d}(\mu, x) \geqslant \beta\}$ and $E_\beta = \{x \in \operatorname{supp} \mu : \bar{d}(\mu, x) = \beta\}$, $\alpha \leqslant \beta \leqslant 2\alpha$.

Lemma 4.1. *There exists a positive constant c such that almost surely for all Borel set* $E \subset [0,1]$, *for all measure function* ψ,

$$\phi - m(X(E)) \geqslant \psi - m(E),$$

where $\phi(t) = \psi(ct^\alpha (\log 1/t)^2)$, X *is defined as in* Eq. (2.2).

Proof. Using Lemma 3.1 and modifying the proof of Theorem 3.1 in Perkins and Taylor (1987) we obtain this lemma immediately.

Theorem 4.1. $\dim C_\beta \geqslant (2\alpha - B)/\gamma$ *a.s., where* $\alpha \leqslant \beta \leqslant 2\alpha$, $\gamma = \beta/\alpha$.

We only prove the theorem in the case that $2 > \gamma > 1$ (the other two cases are trivial because of Theorem 3.2). In order to prove this theorem, it is sufficient to prove that there exists a Borel set $T_\gamma \subset [0,1]$ such that $\psi - m(T_\gamma) = \infty$, where $\psi(s) =$

$s^{2/\gamma-1}(\log 1/s)^{\varepsilon_0}$ (ε_0 is to be selected later) and $X(T_\gamma)$ is a set contained in C_β, which gives $\phi - m(X(T_\gamma)) = \infty$, $\phi(s) = s^{(2\alpha-\beta)/\gamma}(\log 1/s)^\delta$, $\delta = 2(2/\gamma - 1) + 2\varepsilon_0$. Furthermore, one can see that the set T_γ we will construct is a γ-thin set. This technique was first introduced in Perkins and Taylor (1998).

We now construct T_γ. Take $\rho_0 = 1/2$, $\rho_n = \exp(-\rho_{n-1}^{-2/b})$, $n \geqslant 1$, b will be selected later. Let

$$\eta_n = \frac{\rho_n}{(\log 1/\rho_n)^b}, \tag{4.1}$$

$$\Lambda_n = \left\{ I_{j,n} = [j\eta_n, (j+2)\eta_n),\ j = 0, 1, \ldots, \left[\frac{1}{\eta_n}\right] \right\}, \quad n \geqslant 1. \tag{4.2}$$

We say $[j\eta_n, (j+2)\eta_n)$ is a type K interval if it satisfies: there exist S_1, S_2 such that $(j-1)\eta_n < S_1 < j\eta_n$, $(j+2)\eta_n < S_2 < (j+3)\eta_n$ and $|X(s) - X(t)| < \rho_n^{1/\alpha}$, $|X(u) - X(t)| > \rho_n^{1/\alpha\gamma}$, $|X(s) - X(v)| > \rho_n^{1/\alpha\gamma}$, for $u > S_2$, $v \leqslant S_1$, $s \leqslant t$, $s, t \in (S_1, S_2]$. Given a point t in a type K interval, if we denote $r_n = \rho_n^{1/\alpha\gamma}$, we then have

$$K(\omega) \cap (B(x, r_n) - B(x, r_n^\gamma)) = \emptyset, \quad x = X(t). \tag{4.3}$$

Let $M_{n+1}(I)$ be the number of all the type K intervals contained in I, $I \in \Lambda_n$. We are going to show that for a.s. ω, $M_{n+1}(I) \geqslant U_{n+1}$, $\forall I \in \Lambda_n$, n is large enough, and $U_{n+1} \to \infty$ as $n \to \infty$.

Lemma 4.2. *Let $B_j = [j\eta_n, (j+1)\eta_n)$, $j = 1, 2, \ldots, k$, then*

$$P(v(B_j) > \rho_n^{1/\alpha\gamma},\ j = 1, \ldots, k) \approx \rho_n^{k(1-1/\gamma)} \left(\log \frac{1}{\rho_n}\right)^{-kb}. \tag{4.4}$$

Proof. We notice that

$$\left\{ \sum_{i \leqslant K_n} a_i I_{\{\xi_i \in B_j\}} > \rho_n^{1/\alpha\gamma},\ j = 1, \ldots, k \right\} \subset \{v(B_j) > \rho_n^{1/\alpha\gamma},\ j = 1, \ldots, k\},$$

where $K_n = \max\{m: a_m > \rho_n^{1/\alpha\gamma}\}$, and

$$\begin{aligned} P&\left(\sum_{i \leqslant K_n} a_i I_{\{\xi_i \in B_j\}} > \rho_n^{1/\alpha\gamma},\ j = 1, \ldots, k \right) \\ &\geqslant P\left(\bigcap_{j=1}^{k} \left(\bigcup_{m=1}^{K_n} \{\omega: \omega_m \in B_j,\ \omega_s \in [0,1] - B_j,\ s \in \{1, \ldots, K_n\} - \{m\}\} \right) \right) \\ &= \begin{pmatrix} K_n \\ 1 \end{pmatrix} \cdots \begin{pmatrix} K_n - k + 1 \\ 1 \end{pmatrix} \eta_n^k (1 - \eta_n)^{K_n - k} \\ &= K_n \cdots (K_n - k + 1) \eta_n^k (1 - \eta_n)^{K_n - k}, \end{aligned} \tag{4.5}$$

where

$$\begin{pmatrix} m \\ n \end{pmatrix} = \frac{m\cdots(m-n+1)}{n!}$$

but $K_n \approx \rho_n^{-1/\gamma}$, which is less than η_n^{-1}, so $(1-\eta_n)^{K_n-k} > 1/2$ (when n is large), thus the probability is greater than $c_1\rho_n^{k(1-1/\gamma)}(\log 1/\rho_n)^{-kb}$, $c_1 > 0$ is a constant.

As for the other direction we first define $T_n = \max\{m\colon a_m > (1/2)\rho_n^{1/\alpha\gamma}\}$ $(T_n \approx \rho_n^{-1/\gamma})$ and $\mathscr{F} = \{A\colon A \subset \{1,\dots,k\},\ A \neq \{1,\dots,k\}\}$.

One can see that

$$\begin{aligned}
&P(v(B_j) > \rho_n^{1/\alpha\gamma},\ j=1,\dots,k)\\
&\quad\leqslant P\left(\sum_{i\leqslant T_n} a_i I_{\{\xi_i\in B_j\}} > \frac{1}{2}\rho_n^{1/\alpha\gamma},\ j=1,\dots,k\right)\\
&\qquad + P\left(\sum_{A\in\mathscr{F}}\sum_{i\leqslant T_n} a_i I_{\{\xi_i\in B_j\}} > \frac{1}{2}\rho_n^{1/\alpha\gamma},\ j\in A;\right.\\
&\qquad\quad \left.\sum_{i>T_n} a_i I_{\{\xi_i\in B_j\}} > \frac{1}{2}\rho_n^{1/\alpha\gamma},\ j\in\{1,\dots,k\}-A\right)\\
&\quad\equiv N_1 + N_2.
\end{aligned}$$

Note that

$$\left\{\sum_{i\leqslant T_n} a_i I_{\{\xi_i\in B_j\}} > \frac{1}{2}\rho_n^{1/\alpha\gamma},\ j=1,\dots,k\right\} \subset \bigcup_{1\leqslant i_1,\dots,i_k\leqslant T_n} \{\omega_{i_j}\in B_j,\ j=1,\dots,k\},$$

we obtain that

$$N_1 \leqslant \begin{pmatrix} T_n \\ 1 \end{pmatrix}\cdots\begin{pmatrix} T_n-k+1 \\ 1 \end{pmatrix}\eta_n^k \leqslant c_2(\eta_n^{-1/\gamma})^k\eta_n^k = c_2\rho_n^{k(1-1/\gamma)}\left(\log\frac{1}{\rho_n}\right)^{-kb},$$

where $c_2 > 0$ is a constant only depending on k.

By the proof of Lemma 2.1 we know that for any j,

$$P\left(\sum_{i>T_n} a_i I_{\{\xi_i\in B_j\}} > \frac{1}{2}\rho_n^{1/\alpha\gamma}\right) < \exp(-c_3\rho_n^{-(1-1/\gamma)1/\alpha}), \tag{4.6}$$

where $c_3 > 0$ is a constant. So $N_2 \leqslant c_4\rho_n^{k(1-1/\gamma)}(\log 1/\rho_n)^{-kb}$ ($c_4 > 0$ is a constant), which completes the lemma. □

Corollary 4.1. *Define $A_{j,n} = \{I_{j,n}\in\Lambda_n$ is a type K interval$\}$, then $P(A_{j,n}) \approx \rho_n^{2(1-1/\gamma)}$ $(\log 1/\rho_n)^{-2b}$, Λ_n is the same as in* Eq. (4.2).

Proof. In fact

$$A_{j,n} \subset \{X((j+3)\eta_n) - X((j+2)\eta_n) > \rho_n^{1/\alpha\gamma}, X(j\eta_n) - X((j-1)\eta_n) > \rho_n^{1/\alpha\gamma}\},$$

where $X(t) = v[0,t)$. Therefore, by Lemma 4.2, $P(A_{j,n}) \leqslant c\eta_n^{2(1-1/\gamma)}$, $c > 0$ is a constant.

For the other side, given $0<\varepsilon<1/4$,

$$\begin{aligned}A_{j,n} &\supset \{X((\varepsilon+j+2)\eta_n)-X((j-\varepsilon)\eta_n)<\rho_n^{1/\alpha},\\ &\quad X((j-\varepsilon)\eta_n)-X((j-1)\eta_n)>\rho_n^{1/\alpha\gamma}, X((j+3)\eta_n)\\ &\quad -X((j+2+\varepsilon)\eta_n)>\rho_n^{1/\alpha\gamma}\}\\ &\supset \{X((\varepsilon+j+2)\eta_n)-X((j-\varepsilon)\eta_n)<\rho_n^{1/\alpha},\\ &\quad \exists k_1,k_2\leqslant K_n, (j-1)\eta_n<\omega_{k_1}<(j-\varepsilon)\eta_n, (\varepsilon+j+2)\eta_n<\omega_{k_2}<(j+3)\eta_n\}\\ &\equiv B_{j,n},\end{aligned}$$

where $K_n=\max\{m\colon a_m>\rho_n^{1/\alpha\gamma}\}$.

Set $K_n'=\max\{m\colon a_m\geqslant\rho_n^{1/\alpha}\}$. Using Eq. (4.6) we have

$$\begin{aligned}&P(X((j+2+\varepsilon)\eta_n)-X((j-\varepsilon)\eta_n)\geqslant\rho_n^{1/\alpha},\\ &\quad \exists k_1,k_2\leqslant K_n, (j-1)\eta_n<\omega_{k_1}<(j-\varepsilon)\eta_n, (j+2+\varepsilon)\eta_n<\omega_{k_2}<(j+3)\eta_n)\\ &\quad \leqslant c_1\binom{K_n}{1}\binom{K_n-1}{1}\binom{K_n'-2}{1}\eta_n^3\\ &\quad \leqslant c_2\rho_n^{2(1-1/\gamma)}\left(\log\frac{1}{\rho_n}\right)^{-2b}\rho_n^{-1}\eta_n, (K_n\approx\rho_n^{-1/\gamma}, K_n'\approx\rho_n^{-1}),\end{aligned}\tag{4.7}$$

where $c_1,c_2>0$ are constants, and

$$\begin{aligned}&P(\exists k_1,k_2\leqslant K_n, (j-1)\eta_n<\omega_{k_1}<(j-\varepsilon)\eta_n,\\ &\quad (j+2+\varepsilon)\eta_n<\omega_{k_2}<(j+3)\eta_n)\\ &\quad \geqslant\binom{K_n}{1}\binom{K_{n-1}}{1}(1-\varepsilon)^2\eta_n^2(1-(1-\varepsilon)\eta_n)^{K_n-2}\\ &\quad \geqslant c_3\rho_n^{2(1-1/\gamma)}\left(\log\frac{1}{\rho_n}\right)^{-2b},\end{aligned}\tag{4.8}$$

where $c_3>0$ is a constant. So Eqs. (4.7) and (4.8) imply

$$P(B_{j,n})\geqslant c_4\rho_n^{2(1-1/\gamma)}\left(\log\frac{1}{\rho_n}\right)^{-2b},$$

thus we have proved this corollary. □

Let $\mathscr{F}_n$ be the class of all type K intervals in Λ_n.

Lemma 4.3. *If we denote $M_{n+1}(I)$ as the number of type K intervals of $\mathscr{F}_n$ contained in $I\in\Lambda_n$, then there exists a constant $c>0$ such that for a.s. ω,*

$$M_{n+1}(I)\geqslant U_{n+1}\equiv c\rho_{n+1}^{1-2/\gamma}\left(\log\frac{1}{\rho_{n+1}}\right)^{-(3/2)b}\left(\log\frac{1}{\rho_n}\right)^{-b},\quad \forall I\in\Lambda_n,\tag{4.9}$$

where $n\geqslant i_0-1=i_0(\omega)-1$.

Proof. By Corollary 4.1

$$EM_{n+1}(I) \approx \frac{\eta_n}{\eta_{n+1}} \rho_{n+1}^{2(1-1/\gamma)} \left(\log \frac{1}{\rho_{n+1}} \right)^{-2b}$$
$$\geqslant 4U_{n+1},$$

for a suitable choice of c in Eq. (4.9).

Now we estimate $\mathrm{Var}(M_{n+1}(I))$. Let $A_{j,n+1}, K_n, K_n'$ be defined as in Corollary 4.1. We choose ε in Corollary 4.1 to be $K_{n+1}'\eta_{n+1}$. Then from Eqs. (4.7) and (4.8) we obtain that

$$P(A_{j,n+1}) \geqslant K_{n+1}(K_{n+1}-1)(1-K_{n+1}'\eta_{n+1})^2\eta_{n+1}^2(1-\eta_{n+1}+K_{n+1}'\eta_{n+1}^2)^{K_{n+1}-2}$$
$$-c_1 K_{n+1}(K_{n+1}-1)(K_{n+1}'-2)\eta_{n+1}^3(1-\eta_{n+1})^{K_{n+1}-3}.$$

$c_1 > 0$ is the same constant as in Eq. (4.7). Set $t_{n+1} = K_{n+1}(K_{n+1}-1)\eta_{n+1}^2(1-\rho_{n+1}+K_n'\eta_{n+1}^2)^{K_{n+1}-2}$ and $s_{n+1} = K_{n+1}(K_{n+1}-1)\eta_{n+1}^2$. Note $K_{n+1}'\eta_{n+1} \approx (\log 1/\rho_{n+1})^{-b}, (1-\eta_{n+1})^{K_{n+1}} \geqslant 1 - K_{n+1}\eta_{n+1}$ and $K_{n+1}\eta_{n+1} \approx \rho_{n+1}^{1-1/\gamma}(\log 1/(\rho_{n+1}))^{-b}$ we have

$$P(A_{j,n+1}) \geqslant t_{n+1}\left[(1-K_{n+1}'\eta_{n+1})^2 - \frac{c_1(K_{n+1}'-2)\eta_{n+1}}{1-\eta_{n+1}+K_{n+1}'\eta_{n+1}^2}\right]$$
$$\geqslant t_{n+1}(1-(3+2c_1)K_{n+1}'\eta_{n+1})$$
$$= s_{n+1}\left(1-(3+2c_1)\left(\log\frac{1}{\rho_{n+1}}\right)^{-b}\right)(1-\eta_{n+1}+K_{n+1}'\eta_{n+1})^{K_{n+1}-1}$$
$$\geqslant s_{n+1}\left(1-(3+2c_1)\left(\frac{1}{\rho_{n+1}}\right)^{-b}\right)(1-K_{n+1}\eta_{n+1})$$
$$\geqslant s_{n+1}\left(1-(3+2c_1)\left(\log\frac{1}{\rho_{n+1}}\right)^{-b}\right), \quad n \text{ is large enough.}$$

But when $j-k \geqslant 4$ Lemma 4.2 gives

$$P(A_{j,n+1} \cap A_{k,n+1}) \leqslant K_{n+1}(K_{n+1}-1)(K_{n+1}-2)(K_{n+1}-3)\eta_{n+1}^4 + \delta_{n+1},$$
$$\leqslant s_{n+1}^2 + \delta_{n+1},$$

where $\delta_{n+1} < \eta_{n+1}^5$.

The above two inequalities imply that when $j-k>4$,

$$P(A_{j,n+1} \cap A_{k,n+1}) - P(A_{j,n+1})P(A_{k,n+1}) \leqslant (3+2c_1)s_{n+1}^2\left(\log\frac{1}{\rho_{n+1}}\right)^{-b} + \delta_{n+1}. \tag{4.10}$$

Using Eq. (4.10) we have

$$EM_{n+1}(I)^2 - (EM_{n+1}(I))^2 \leqslant \sum_{|j-k|\leqslant 4}[EI_{A_{j,n+1}}I_{A_{k,n+1}} - EI_{A_{j,n+1}}EI_{A_{k,n+1}}]$$
$$+ \sum_{|j-k|>4}[EI_{A_{j,n+1}}I_{A_{k,n+1}} - EI_{A_{j,n+1}}EI_{A_{k,n+1}}]$$

$$\leqslant c_2\, EM_{n+1}(I) + c_2' \left(\log \frac{1}{\rho_{n+1}}\right)^{-b} (EM_{n+1}(I))^2$$
$$+\left(\frac{\eta_n}{\eta_{n+1}}\right)^2 \delta_{n+1},$$

where $c_2 > 0, c_2'$ are constants.

By Chebyshev's inequality one obtains

$$P(M_{n+1}(I) < U_{n+1}) \leqslant c_3 \left(\log \frac{1}{\rho_{n+1}}\right)^{-(2/3)b}, \quad c_3 > 0 \text{ is a constant.}$$

Since the above I is an arbitrary interval in Λ_n, so

$$P\left(\cup_{I\in\Lambda_n}\{M_{n+1}(I) < U_{n+1}\}\right) \leqslant \frac{c_3}{\eta_n}\left(\log \frac{1}{\rho_{n+1}}\right)^{-(2/3)b}.$$

Choose $0 < b < 1$; we are going to show

$$\sum_{n\geqslant 1} \frac{c_3}{\eta_n}\left(\log \frac{1}{\rho_{n+1}}\right)^{-(2/3)b} < \infty.$$

In fact, when n is large,

$$\frac{1}{\eta_n}\left(\log \frac{1}{\rho_{n+1}}\right)^{-(2/3)b} = \rho_n^{1/3}\left(\log \frac{1}{\rho_n}\right)^{b} < \rho_n^{1/4} \quad (\text{since } \rho_n \downarrow 0),$$

but $\rho_1 = \exp(-(1/2)^{-2/b}) < \mathrm{e}^{-4}$, $\rho_2 \leqslant \exp(-(\mathrm{e}^{-4})^{-2/b}) \leqslant \exp(-\mathrm{e}^{4\times 2}) < \mathrm{e}^{-2^3}$, for $b<1$. If we assume that $\rho_n \leqslant \exp(-2^{n+1})$, then $\rho_{n+1} \leqslant \exp(-(\mathrm{e}^{-2^{n+1}})^{-2/b}) < \exp(-\mathrm{e}^{2^{n+1}\cdot 2}) \leqslant \mathrm{e}^{-2^{n+2}}$. So $(1/\eta_n)(\log 1/\rho_{n+1})^{-b} \leqslant (\mathrm{e}^{-2^{n+1}})^{1/4} < 1/2^n$, when n is large, which implies that $\sum_{n\geqslant 1}(1/\eta_n)(\log 1/\rho_{n+1})^{-2/3b} < \infty$.

Borel–Cantelli lemma tells us that for a.s. ω, there exists $i_0 = i_0(\omega)$ such that when $n \geqslant i_0 - 1$,

$$M_{n+1}(I) \geqslant U_{n+1}, \quad \forall I \in \Lambda_n. \tag{4.11}$$

Now we can construct the Cantor like set T_γ. Given $\omega \in \Omega_0 \equiv \{\omega\colon \exists i_0 = i_0(\omega)$ such that Eq. (4.11) is true$\}$. Choose U_{i_0} type K intervals from $\mathscr{F}_{i_0}$, say $I_{1,i_0},\dots,I_{U_{i_0},i_0}$ (the class of this set is denoted as $\mathscr{G}_{i_0}$). Let $F_1 = \bigcup_{j=1}^{U_{i_0}} I_{j,i_0}$. For each $1 \leqslant j \leqslant U_{i_0}$, choose U_{i_0+1} type K intervals from I_{j,i_0}, denote all of them as $\mathscr{G}_{i_0+1} \equiv \{I_{1,i_0+1},\dots,I_{U_{i_0}U_{i_0+1},i_0+1}\}$. Let $F_2 = \bigcup_{j=1}^{U_{i_0}U_{i_0+1}} I_{j,i_0+1}$. Continuing this process we obtain a sequence of sets $F_1, F_2, \dots$. Define

$$T_\gamma(\omega) = \bigcap_{n\geqslant 1} F_n(\omega), \ \omega\in\Omega_0; \quad T_\gamma = [0,1], \ \omega \in \Omega_0^c. \tag{4.12}$$

We know that T_γ is not empty and Eq. (4.3) tells us T_γ is a γ-thin set a.s. Furthermore, given $x = X(t) \in X(T_\gamma)$, let $r_n = \rho_n^{1/\alpha\gamma}$, we have

$$\mu B(X(t), r_n) \leqslant 3\rho_n = 3r_n^{\beta},$$

so

$$\limsup_{n\to 0} \frac{\log \mu B(x, r_n)}{\log r_n} \geqslant \beta,$$

thus $X(T_\gamma)\subset C_\beta \equiv \{x\in \operatorname{supp}\mu\colon \bar{d}(\mu,x)\geqslant\beta\}$. Now we have proved $X(T_\gamma)$ is a non-empty γ-thin set contained in C_β.

Let $\psi(s)=s^{2/\gamma-1}(\log 1/s)^{\varepsilon_0}$. We will show $\psi-m(T_\gamma)=\infty$ a.s.

Define a sequence of random measures $\{\mu_i, i\geqslant 0\}$ on [0,1]: For $\omega\in\Omega_0, \mu_0$ is the unit measure on $[0,1], \mu_1(I_{j,i_0})=1/U_{i_0},\ j=1,\dots,U_{i_0},\dots,\mu_k(I_{j,i_0+k})=1/(U_{i_0}\cdots U_{i_0+k}),\ j=1,\dots,U_{i_0}U_{i_0+1}\cdots U_{i_0+k}$, define $\bar{\mu}=\lim_{k\to\infty}\mu_k$; $\bar{\mu}\equiv 1$ if $\omega\in\Omega_0^c$. Let

$$I_\psi(\bar{\mu})=\int\int_{T_\gamma\times T_\gamma}\frac{1}{|x-y|^{2/\gamma-1}(\log(1/|x-y|))^{\varepsilon_0}}\bar{\mu}(\mathrm{d}x)\bar{\mu}(\mathrm{d}y)$$

and

$$Y=\int\int_{\{(x,y)\in T_\gamma\times T_\gamma:|x-y|\geqslant\eta_{i_0}\}}\frac{1}{|x-y|^{2/\gamma-1}(\log(1/|x-y|))^{\varepsilon_0}}\bar{\mu}(\mathrm{d}x)\bar{\mu}(\mathrm{d}y).$$

For each $\omega\in\Omega_0, Y(\omega)<\infty$. Now set

$$\begin{aligned}A&=\bigcup_{i\geqslant i_0}\bigcup_{j=1}^{n_i}\{(x,y)\in T_\gamma\times T_\gamma\colon 2^{j+1}\eta_i\geqslant|x-y|>2^j\eta_i\}\\&\equiv\bigcup_{i\geqslant i_0}\bigcup_{j=1}^{n_i}G_{j,i},\end{aligned}$$

n_i satisfies $2^{n_i+1}\eta_i\geqslant\eta_{i-1}\geqslant 2^{n_i}\eta_i,\ i\geqslant i_0$.

Note $G_{j,i}$ can be covered by at most $c_1 2^j N_i (N_i\equiv U_{i_0}U_{i_0+1}\cdots U_i,\ c_1>0$ is a constant) product sets in the form of $I_{k,i}\times I_{k',i}, I_{k.i}, I_{k',i}\in\mathscr{G}_i$ and the distance between $I_{k,i}$ and $I_{k',i}$ is less than $2^{j+1}\eta_i$. Hence

$$\begin{aligned}&\int\int_{G_{j,i}}\frac{1}{|x-y|^{2/\gamma-1}(\log(1/|x-y|))^{\varepsilon_0}}\bar{\mu}(\mathrm{d}x)\bar{\mu}(\mathrm{d}y)\\&\quad\leqslant\frac{c_1N_i2^j}{(N_i)^2}\frac{1}{(2^j\eta_i)^{2/\gamma-1}(\log(1/2^{j+1}\eta_i))^{\varepsilon_0}},\end{aligned}$$

therefore for $\omega\in\Omega_0$,

$$\begin{aligned}I_\psi(\bar{\mu})-Y&\leqslant c_1\sum_{i\geqslant i_0}\sum_{j=1}^{n_i}\frac{2^j}{N_1}\cdot\frac{1}{(2^j\eta_i)^{2/\gamma-1}(\log(1/2^j\eta_i))^{\varepsilon_0}}\\&\leqslant c_2\sum_{i\geqslant i_0}\frac{\eta_{i-1}/\eta_i}{N_i}\frac{1}{\eta_i^{2/\gamma-1}(\log 1/\eta_i)^{\varepsilon_0}}\\&\leqslant c_2\sum_{i\geqslant i_0}\left(\log\frac{1}{\eta_i}\right)^{-\varepsilon_0}\quad(N_i>U_{i-1}U_i)\\&<\infty,\end{aligned}$$

if we choose $\varepsilon_0>0$. This implies $I_\psi(\mu)<\infty$ a.s., so $\psi-m(T_\gamma)=\infty$ a.s. Finally we have finished the proof of Theorem 4.1.

An easy argument gives:

Corollary 4.2. *Let* $E_\beta=\{x\in\operatorname{supp}\mu\colon \bar{d}(\mu,x)=\beta\}$, *then* $\dim E_\beta\geqslant(2\alpha-\beta)/\gamma,\ \alpha\leqslant\beta\leqslant 2\alpha$ *and* $\gamma=\beta/\alpha$.

According to Lemma 2.3 we have

Corollary 4.3. $\mathrm{Dim}(C_\beta) \geqslant \mathrm{Dim}(E_\beta) \geqslant 2\alpha - \beta$ *a.s., for* $\alpha \leqslant \beta \leqslant 2\alpha$.

5. The upper bound of Hausdorff dimension of the set of points where upper local dimensions equal to $\beta(\alpha \leqslant \beta \leqslant 2\alpha)$

Lemma 5.1. *Given* $0 \leqslant u_1 < v_1 \leqslant u_2 < v_2 \leqslant \cdots < v_k \leqslant a < 1$ *with* $v_i - u_i = 2^{-n(\alpha+\varepsilon)}$ *and* $0 \leqslant s_i < r_i \leqslant 1,\ i = 1, \ldots, k$, *then*

$$P(v(a, a + 2^{-n(\alpha+\varepsilon)}] > 2^{-n}, v(u_i, v_i] \in [s_1, r_1), i = 1, \ldots, k)$$
$$\leqslant cP(v(a, a + 2^{-n(\alpha+\varepsilon)}] > 2^{-n})P(v(u_i, v_i] \in [s_i, r_i), i = 1, \ldots, k),$$

$c > 0$ *is a constant.*

Proof. Define T_1 as

$$T_1 = \inf\left\{m: \sum_{i>m} a_i \leqslant 2^{-n-1}\right\}.$$

Let $q_n = \max\{k: a_k > 2^{-n-1}\}$, define

$$A_1 = \{1\}, \ldots, A_{q_n} = \{q_n\}, A_{q_n+1} = \{q_n + 1, \ldots, q_n + t_1\},$$

$$A_{q_n+2} = \{q_n + t_1 + 1, \ldots, q_n + t_2 + t_1\}, \ldots,$$

where $t_1 = \inf\{k: \sum_{i=1}^k a_{q_n+i} > 2^{-n-1}\}$, $t_2 = \inf\{k: \sum_{i=1}^k a_{q_n+t_1+i} > 2^{-n-1}\}, \ldots$. There must be a unique positive integer T such that

$$\sum_{j\in A_i} a_j > 2^{-n-1} \quad \text{for all } i \leqslant T,$$

and

$$\sum_{j>\max A_T} a_j \leqslant 2^{-n-1}.$$

Obviously $T_2 \equiv \max A_T \geqslant T_1$.

Let $\mathscr{F} = \{B$: B is a set of finite many positive such that $\max(B) \leqslant \max(A_T)$ and $\sum_{i\in B} a_i > 2^{-n-1}\}$. Denote $\bar{T}$ as the number of all the elements in $\mathscr{F}$. Rename sets in $\mathscr{F} - \{A_1, \ldots, A_T\}$ as $\{A_{T+1}, \ldots, A_{\bar{T}}\}$.

Let $\mathscr{G}_m = \{B$: B is a set of finite many integers such that $\sum_{j\in B} a_j \in [s_m, r_m)$ and $\sum_{j>\max(B)} a_j + \sum_{j\in B} a_j < r_m\} \equiv \{B_{m,j}: j = 1, \ldots\}$, $m = 1, \ldots, k$.

By the definition of T one can see that

$$\{v(a, a + 2^{-n(\alpha+\varepsilon)}] > 2^{-n}\} \subset \left\{\sum_{i=1}^{T_2} a_i I_{\{\xi_i \in (a, a+2^{-n(\alpha+\varepsilon)}]\}} > 2^{-n-1}\right\}$$
$$= \bigcup_{j=1}^{\bar{T}} \{\omega: \xi_i(\omega) \in (a, a + 2^{-n(\alpha+\varepsilon)}], i \in A_j\}$$

$$\equiv \bigcup_{j=1}^{\bar{T}} C_j = \bigcup_{j=1}^{\bar{T}} D_j, \tag{5.1}$$

where $D_1 = C_1,\ D_j = C_j - \bigcup_{i=1}^{j-1} C_i,\ j \geqslant 2.$

It is also easy to see that

$$\{v(u_m, v_m] \in [s_m, r_m)\} = \bigcup_{j \geqslant 1} \{\omega\colon\ \xi_i(\omega) \in (u_m, v_m],\ i \in B_{j,m};$$
$$\xi_i(\omega) \in [0,1] - (u_m, v_m],\ i \in \{1, \dots, \max(B_{j,m})\} - B_{j,m}\}, \tag{5.2}$$

where $m = 1, \dots, k$.

In fact, when $j \leqslant T$,

$$D_j = \{\omega\colon\ \xi_k(\omega) \in (a, a + 2^{-n(\alpha+\varepsilon)}], k \in A_j\}$$
$$\cap \bigcap_{i=1}^{j-1} (\{\omega\colon \text{ there exists at least one } k \in A_i,\ \xi_k(\omega) \in [0,1] - (a, a + 2^{-n(\alpha+\varepsilon)}]\}). \tag{5.3}$$

Note that $A_i \cap A_{i'} = \emptyset$ $(i \neq i', i, i' \leqslant j)$ and the independence of $\{\xi_k\}_{k \geqslant 1}$, then if we set $d_n = 2^{-n(\alpha+\varepsilon)}$, we have

$$P(D_j) = d_n^{\#(A_j)} (1 - d_n^{\#(A_1)}) \cdots (1 - d_n^{\#(A_{j-1})}), \tag{5.4}$$

where $\#(A_j) =$ the number of integers in A_j.

From Eq. (5.2) we see that for each $1 \leqslant m \leqslant k$ we can write $\{v(u_m, v_m] \in [s_m, r_m)\}$ as $\bigcup_{j \geqslant 1} S_{j,m}$, where $S_{j,m} \cap S_{j',m} = \emptyset$, $\forall j \neq j'$ and

$$S_{j,m} = F_{k_1}^{(m)} \cap \cdots \cap F_{k_t}^{(m)}, \quad t \text{ is an integer}, \tag{5.5}$$

where $F_{k_i}^{(m)} = \{\omega\colon\ \xi_{k_i}(\omega) \in (u_m, v_m])\}$ or $\{\omega\colon\ \xi_{k_i}(\omega) \in [0,1] - (u_m, v_m]\}$ and satisfy that if we let $\Sigma(S_{j,m}) = \{r = k_1, \dots, k_t\colon\ \xi_r(\omega) \in (u_m, v_m],\ \forall \omega \in S_{j,m}\}$, then $\Sigma(S_{j,m}) \in \mathscr{G}_m$. For convenience, we set $\Lambda(S_{j,m}) = \{k_1, \dots, k_t\} - \Sigma(S_{j,m})$. Now given $(j_1, \dots, j_k)$, let $S = \prod_{m=1}^{k} S_{j_m,m}$ and $\Sigma(S) = \bigcup_{m=1}^{k} \Sigma(S_{j_m,m}), \Lambda(S) = \bigcup_{m=1}^{k} \Lambda(S_{j_m,m})$. Without loss of generality, we assume that $(\Sigma(S_{j_m,m}) \cup \Lambda(S_{j_m,m}) \cap \Sigma(S_{j_{m'},m'}) \cup \Lambda(S_{j_{m'},m'})) = \emptyset$, $\forall m \neq m'$. So we have

$$P(S) = d_n^{\#(\Sigma(S))} (1 - d_n)^{\#(\Lambda(S))}. \tag{5.6}$$

Combining Eqs. (5.4) and (5.6) one has

$$P(D_j)P(S) = d_n^{\#(A_j) + \#(\Sigma(S))} \prod_{i=1}^{j-1} [(1 - d_n^{\#(A_i)})(1 - d_n)^{\#(A_i \cap \Lambda(S))}](1 - d_n)^{\#(\Lambda(S) - \cup_{i=1}^{j-1} A_i)}. \tag{5.7}$$

But if $D_j \cap S \neq \emptyset$,

$$P(D_j \cap S) = d_n^{\#(A_j) + \#(\Sigma(S))}$$
$$\times \prod_{i=1}^{j-1} \left\{ \sum_{t=0}^{\#(A_i \cap \Lambda(S)) - 1} \binom{\#(A_i \cap \Lambda(S))}{t} d_n^t (1 - 2d_n)^{\#(A_i \cap \Lambda(S)) - t} \right.$$

$$+ d_n^{\#(A_i\cap\Lambda(S))}\left(\sum_{t=0}^{\#(A_i-\Sigma(S)-\Lambda(S))-1}\binom{\#(A_i-\Sigma(S)-\Lambda(S))}{t}\right.$$

$$\left. d_n^t(1-d_n)^{\#(A_i-\Sigma(S)-\Lambda(S))-t}\right)\Big\}\,(1-d_n)^{\#(\Lambda(S)-\cup_{i=1}^{j-1}A_i)}$$

$$= d_n^{\#(A_j)+\#(\Sigma(S))}\prod_{i=1}^{j-1}[(1-d_n)^{\#(A_i\cap\Lambda(S))}-d_n^{\#(A_i-\Sigma(S)}]$$

$$(1-d_n)^{\#(\Lambda(S)-\cup_{i=1}^{j-1}A_i)}. \tag{5.8}$$

Note that for any $i<j$,

$$(1-d_n^{\#(A_i)})(1-d_n)^{\#(A_i\cap\Lambda(S))}\geqslant(1-d_n)^{\#(A_i\cap\Lambda(S))}-d_n^{\#(A_i-\Sigma(S))},$$

thus for any $j\leqslant T$, we always have

$$P(D_j\cap S)\leqslant P(D_j)P(S).$$

When $j>T$,

$$D_j\subset C_j-\bigcup_{i=1}^{T}C_i = D_j\cap\Big\{\omega\in C_j-\bigcup_{i=1}^{T}C_i\colon\ \exists T<t\leqslant j-1 \text{ such that}$$

$$X_l(\omega)\in[a,a+2^{-n(\alpha+\varepsilon)}),\ \forall l\in A_t\Big\}.$$

Observe Eq. (5.3) we have $P(D_j)\geqslant 1/2P(C_j-\bigcup_{i=1}^{T}C_i)$. Modifying the above method we obtain that

$$P\left(\left(C_j-\bigcup_{i=1}^{T}C_i\right)\cap S\right)\leqslant P\left(C_j-\bigcup_{i=1}^{T}C_i\right)P(S),$$

this implies that $P(S\cap D_j)\leqslant 2P(S)P(D_j)$, which completes the lemma.

Remark. In fact, the above lemma remains true (the constant c may not be the same) if we change the condition "$v_i-u_i=2^{-n(\alpha+\varepsilon)}$" to "$v_i-u_i\leqslant 2^{-n(\alpha+\varepsilon)}$".

Recalling that $S=\prod_{m=1}^{k}S_{j_m,m}$ we set $\delta_m=v_m-u_m$, $W_m=\#(\Sigma(S_{j_m,m}))$, $K_{l,m}=\#(A_l\cap\Lambda(S_{j_m,m}))$, $1\leqslant l\leqslant j$, $1\leqslant m\leqslant k$. From the calculations of Eq. (5.8) we similarly obtain for $j\leqslant T$,

$$P(D_j\cap S)\leqslant d_n^{\#(A_j)}\delta_1^{W_1}\cdots\delta_k^{W_k}(1-\delta_1)^{K_{1,1}}\cdots(1-\delta_k)^{K_{1,k}}\cdots(1-\delta_1)^{K_{j-1,1}}$$

$$\cdots(1-\delta_k)^{K_{j-1,k}}q,$$

where $q=(1-\delta_1)^{\#(\Lambda(S_{j_1,1})-\cup_{i=1}^{j-1}A_i)}\cdots(1-\delta_k)^{\#(\Lambda(S_{j_k,k})-\cup_{i=1}^{j-1}A_j)}$.

We also have

$$P(D_j)P(S)=d_n^{\#(A_j)}\delta_1^{W_1}\cdots\delta_k^{W_k}$$

$$(1-d_n)^{\#(A_1)}(1-\delta_1)^{K_{1,1}}\cdots(1-\delta_k)^{K_{1,k}}\cdots$$

$$(1-d_n)^{\#(A_{j-1})}(1-\delta_1)^{K_{j-1,1}}\cdots(1-\delta_k)^{K_{j-1,k}}q.$$

Obviously for any $i<j$,

$$(1-\delta_1)^{K_{i,1}}\cdots(1-\delta_k)^{K_{i,k}}\leqslant 2[(1-\delta_1)^{K_{i,1}}\cdots(1-\delta_k)^{K_{i,k}}-d_n^{\#(A_i)}$$

$$(1-\delta_1)^{K_{i,1}}\cdots(1-\delta_k)^{K_{i,k}}).$$

Therefore

$$P(D_j \cap S) \leqslant 2P(D_j)P(S).$$

In the same way as in Lemma 5.1 one can show when $j > T$, $P(D_j \cap S) \leqslant 4P(D_j)P(S)$.

Corollary 5.1. *Given* $\varepsilon > 0$ *and* $a \in [0,1)$ *such that* $a + 2^{-n(\alpha+\varepsilon)} \leqslant 1$, *then*

$$P(v(a, a+2^{-n(\alpha+\varepsilon)}] > 2^{-n}, A) \leqslant cP(v(a, a+2^{-n(\alpha+\varepsilon)}] > 2^{-n})P(A), \tag{5.9}$$

for any $A \in \mathscr{F}_a \equiv \sigma(v[0,t], t \leqslant a)$ (*the* σ*-field generated by* $\{v[0,t], t \leqslant a\}$), $c > 0$ *is a constant.*

Proof. Let $\mathscr{G} = \{B\colon B$ satisfies Eq. (5.9)$\}$. Obviously, $\mathscr{G}$ is a monotone class. In order to prove this corollary, we only need to show that Eq. (5.9) is true for any $B \in \mathscr{H} =$ the class of unions of finite disjoint sets in form of $\{v[0,t_i] \in B_i,\ i = 1, \ldots, k\}$, where $t_i \leqslant a$, $B_i \in \mathscr{E} \equiv \{$unions of finite disjoint left-semiclosed intervals in $[0,1]\} \cup \{\emptyset, [0,1]\}$, $i = 1, \ldots, k$. In fact, $\mathscr{H}$ is a field. If we can prove that Eq. (5.9) is true for any $B \in \mathscr{H}$, by the monotone class theorem we know every $B \in \mathscr{F}_a$ satisfies Eq. (5.9).

Now given $A = \{v[0,t_1] \in [r_1, s_1), \ldots, v[0,t_k] \in [r_k, s_k)\}$. All we have to do is to show A satisfies Eq. (5.9). In fact, for each i, there exists a unique interger m_i such that

$$m_i 2^{-n(\alpha+\varepsilon)} \leqslant t_i < (m_i + 1)2^{-n(\alpha+\varepsilon)}, \quad i = 1, \ldots, k.$$

Without loss of generality we assume $m_k \geqslant m_{k-1} \geqslant \cdots \geqslant m_1$. So A can be written as

$$\begin{aligned}
&\{v[0,d_n] + \cdots + v((m_1 - 1)d_n, m_1 d_n] + v(m_1 d_n, t_1] \in [s_1, r_1)\} \\
&\quad v[0,d_n] + \cdots + v(m_1 d_n, t_1] + v(t_1, (m_1+1)d_n] + \cdots + v((m_2 - 1)d_n, m_2 d_n] \\
&\quad + v(m_2 d_n, t_2] \in [s_2, r_2), \ldots, v[0,d_n] + \cdots v(m_{k-1} d_n, t_{k-1}] \\
&\quad + \{v(t_{k-1}, (m_{k-1}+1)d_n + \cdots + v(m_k d_n, t_k] \in [s_k, r_k)\}
\end{aligned}$$

where $d_n = 2^{-n(\alpha+\varepsilon)}$.

Let $\mathscr{I} = \{[0,d_n], \ldots, ((m_1 - 1)d_n, m_1 d_n], (m_1 d_n, t_1], \ldots, (m_k d_n, t_k]\}$. Define $\mathscr{H}_0 = \{\{v(I_1) \in B_1, \ldots, v(I_j) \in B_j\}\colon I_i \in \mathscr{I}, B_i \in \mathscr{E}, = 1, \ldots, j,\ j \leqslant \#(\mathscr{I})\}$. Let $\mathscr{H}_1$ be the field generated by $\mathscr{H}_0$. Using Lemma 5.1 we obtain

$$P(v(a, a+2^{-n(\alpha+\varepsilon)}] > 2^{-n}, C) \leqslant cP(v(a, a+2^{-n(\alpha+\varepsilon)}] > 2^{-n})P(C), \quad \forall C \in \mathscr{H}_1.$$

By the monotone class theorem again we know Eq. (5.9) is true for every set in $\sigma(\mathscr{H}_1)$, but the given set A is exactly in this σ-algebra. We have finally completed the proof of this corollary. $\square$

Theorem 5.1. *Let* $\mathscr{F}_t$ *be defined as in Corollary* 5.1. *If* τ *is a finite stopping time respect to* $\{\mathscr{F}_t\}_{t \geqslant 0}$, *then for any* $A \in \mathscr{F}_\tau \equiv \{A \in \mathscr{B}([0,1]^{\mathrm{N}})\colon A \cap \{\tau \leqslant t\} \in \mathscr{F}_t\}$,

$$P(v(\tau, \tau+2^{-n(\alpha+\varepsilon)}] > 2^{-n}, A) \leqslant cP(v(\tau, \tau+2^{-n(\alpha+\varepsilon)}] > 2^{-n})P(A), \tag{5.10}$$

$c > 0$ *is a constant.*

Proof. First note that $\{Z_t := v[0,t],\ t \geqslant 0\}$ is a right continuous process. Using Corollary 5.1 and modifying the method of Theorem 12.42 in Breiman (1968), we obtain this theorem. □

By Theorem 5.1 and Lemma 2.1 we immediately obtain:

Corollary 5.2. *Let* $A = [a,b) \subset [0,1]$ *and* $\eta_A = \inf\{t \in [0,1]\colon v[0,t) \in A\}$. *Then* $P(v(\eta_A, \eta_A + 2^{-n(\alpha+\varepsilon)}] > 2^{-n} \mid \eta_A < \infty) \leqslant c2^{-n\varepsilon}$, $c > 0$ *is a constant.*

Now we have finished the preparations for the following theorem.

Theorem 5.2. *Let* C_β *be defined in* Section 4, *then* $C_\beta = \emptyset$ *a.s., if* $\beta > 2\alpha$.

Proof. Using Corollary 5.2 and Lemma 2.2 and modifying the method in [8] we obtain this theorem. □

Theorem 5.3. $\dim C_\beta \leqslant (2\alpha - \beta)/\gamma$ *a.s., where* $\alpha \leqslant \beta \leqslant 2\alpha$, $\gamma = \beta/\alpha$, C_β *is defined as in* Theorem 5.2.

From the proof of Theorem 4.3 in Hu and Taylor (1997) we know in order to prove this theorem we only need Lemma 2.2 and the following lemma.

Lemma 5.2. *Let* $\tau_i = \inf\{t \in [0,1]\colon v[0,t) \in [k_i/2^n, k_i + 1/2^n)\}$, $k_i + 2 \leqslant k_{i+1}$, $i = 1,\ldots,s$; *then there exists a constant* $c > 0$ *such that*

$$P(v(\tau_i, \tau_{i+1}] > 2^{-n}, \tau_{i+1} - \tau_i \leqslant 2^{-n(\alpha+\varepsilon)},\ i = 0,1,\ldots,s-1) \leqslant c_s 2^{-n\varepsilon s}, \tag{5.11}$$

where $\tau_0 = 0$.

Proof. By Theorem 5.1 and Lemma 2.1 we have

$$\begin{aligned}
&P(v(\tau_i, \tau_{i+1}] > 2^{-n},\ \tau_{i+1} - \tau_i \leqslant 2^{-n(\alpha+\varepsilon)},\ i = 1,\ldots,s-1)\\
&\quad \leqslant c_1 P(\tau_s < \infty) P(v(\tau_i, \tau_{i+1}] > 2^{-n}, \tau_{i+1} - \tau_i \leqslant 2^{-n(\alpha+\varepsilon)},\\
&\qquad\quad i = 1,\ldots,s-2 \mid \tau_s < \infty)\\
&\qquad P(v(\tau_s, \tau_s + 2^{-n(\alpha+\varepsilon)}] > 2^{-n} \mid \tau_s < \infty)\\
&\quad \leqslant (c_2 2^{-n\varepsilon})^s.
\end{aligned}$$

(Note that Eq. (5.2) gives $P(\tau_i < \infty) > 0$, $i = 1,\ldots,s$). □

Combining Theorems 5.3, 4.1 and Corollary 4.2, we have:

Theorem 5.4. $\dim C_\beta = \dim E_\beta = (2\alpha - \beta)/\gamma$ *a.s., where* $\gamma = \beta/\alpha$, $\alpha \leqslant \beta \leqslant 2\alpha$.

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